PMM U.S.S.R.,Vol.50,No.3,pp.255- 260,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

EXTREMAL STRATEGIES IN NON-LINEAR DIFFERENTIAL GAMES*

E.G. AL'BREKHT

A game-theoretic problem of guidance /1, 2/ is studied for non-linear controlled objects in the case when the control domains of the players depend on the phase coordinates. A procedure for constructing the reference functions of the domains of accessibility by the objects in question is described under specified conditions. The conditions under which the reference functions of the domains of accessibility are differentiable with respect to the initial position are given. The results obtained make it possible to use the rule of extremal aiming /1, 2/ to solve the guidance problem. The condition of regularity of the game /1/ is introduced in the usual manner, and is confirmed by finding a solution to a finite-dimensional extremal problem. It is shown that in the regular case the extremal strategies give the game-theoretic guidance problem a saddle point.

1. Formulation of the problem. Consider controlled antagonistic objects described by the equations

$$y^{\cdot} = u \in P(t, y), \quad z^{\cdot} = v \in Q(t, z) \tag{1.1}$$

where y, z are the n-dimensional phase vectors, u, v are n-dimensional vectors of the controlling actions, and $P(t, y), Q(t, z)$ are the domains of control by the players. The game is played over a given time interval $t_0 \leqslant t \leqslant \vartheta$, and the game payoff is given by the relation

$$\gamma[\vartheta] = \sigma(z(\vartheta) - y(\vartheta)) = \sigma(x(\vartheta)) \tag{1.2}$$

where $\sigma(x)$ is a given function of the vector argument $x = z - y$. The first player, who is in charge of the control $u \in P(t, y)$, tries to minimize the quantity $\gamma[\vartheta]$, and the second player, in charge of the control $v \in Q(t, z)$, tries to maximize the quantity $\gamma[\vartheta]$.

We shall assume that at every instant of time t the players know the values of $y[t]$ and $z[t]$, and the controls are formed according to the feedback principle, i.e. the realized values of $u[t]$ and $v[t]$ are formed from the information available concerning the quantities $y[t]$ and $z[t]$.

We will determine the admissible strategies U and V of the players in the form of multi-valued mappings, semicontinuous from above over the inclusion, which place in correspondence with every position $\{t, y, z\}$ the convex sets $U^*(t, y, z) \subset P(t, y)$ and $V^*(t, y, z) \subset Q(t, z)$, and we will regard at the motions the solutions of the corresponding contingent equations /1–3/.

Let $(\gamma[\vartheta]/t_0, y_0, z_0, u, v)$ be the realization of the quantity (1.2) corresponding to the initial position $\{t_0, y_0, z_0\}$ with controls u and v.

Problem 1.1. We require to find, amongst the admissible strategies U, the optimal strategy U° which ensures the inequality

$$(\gamma[\vartheta]/t_0, y_0, z_0, U^0, v) \leqslant \min_U \sup_{v[t]} \inf_{y[t]} (\gamma[\vartheta]/t_0, y_0, z_0, U, v)$$

irrespective of the initial position $\{t_0, y_0, z_0\}$.

Problem 1.2. We require to find, amongst the admissible strategies V, the optimal strategy V°, which ensures that the inequality

$$(\gamma[\vartheta]/t_0, y_0, z_0, u, V^0) \geqslant \max_V \inf_{u[t]} \sup_{z[t]} (\gamma[\vartheta]/t_0, y_0, z_0, u, V)$$

holds irrespective of the initial position $\{t_0, y_0, z_0\}$.

The aim to this paper is to justify the rules of extremal aiming /1, 2/ for solving problems 1.1 and 1.2 for controlled objects of the form (1.1), when the domains of control by the players depend on the phase coordinates.

2. Domain of accessibility. Consider the control system

$$x^{\cdot} = w \in R(t, x) \tag{2.1}$$

*Prikl.Matem.Mekhan.,50,3,339-345,1986

We will assume that the multivalued mapping $R(t, x)$, describing the domain of control, satisfies the following conditions:

1) $R(t, x)$ is a convex, closed and bounded set depending continuously on the position $\{t, x\}$;

2) any vector $w(t, x) \in R(t, x)$ satisfies the inequality

$$\| w(t, x) \| \leqslant c_1 (1 + \| x \|), \ c_1 - \text{const}$$

3) we have the following inclusion for any $\lambda \in [0, 1]$, $x^{(1)}$ and $x^{(2)}$ at almost all $t \in [t_0, \vartheta]$

$$\lambda R(t, x^{(1)}) + (1 - \lambda) R(t, x^{(2)}) \subseteq R(t, \lambda x^{(1)} + (1 - \lambda) x^{(2)})$$

4) irrespective of the value of the non-zero vector ψ, the maximum of the expression

$$\max_{w \in R(t, x)} \psi' w = \psi' w^\circ (\psi, t, x) = \eta [\psi, t, x] \tag{2.2}$$

is attained on the unique vector $w^\circ(\psi, t, x)$, which is continuous in t, x and ψ, and continuously differentiable in x, and

$$\| \partial w^\circ (\psi, t, x) / \partial x \| \leqslant c_2, \ c_2 = \text{const}$$

5) whatever the vector l, the problem

$$\begin{gathered} x^{\cdot} = w^\circ (\psi, t, x), \ x(t_0) = x_0 \\ \psi^{\cdot} = -\left[\frac{\partial w^\circ (\psi, t, x)}{\partial x} \right]' \psi, \psi(\vartheta) = l \end{gathered} \tag{2.3}$$

has a unique solution $\{x^\circ (t/t_0, x_0, l), \psi^\circ (t/t_0, x_0, l)\}$ at all x_0.

Notes. 2.1. Condition 5) holds for the linear systems if condition 4) holds, and for quasilinear systems with additional assumptions /4/.

2.2. In the case of non-linear systems with a control domain independent of the phase coordinates, when the set $R(t, x)$ is defined by the equation

$$R(t, x) = \{w \colon w = f(t, x, u), \ u \in P(t)\}$$

where $P(t)$ is a convex, closed and bounded set, condition 3) can be replaced by the requirement that the reference function $\eta [\psi, t, x]$ of the set $R(t, x)$ must be concave in x.

When conditions 1) and 2) hold, Eq.(2.1), regarded as a differential equation in contingencies has, for any initial position $\{t_0, x_0\}$, generally speaking, a non-unique solution $x(t/t_0, x_0)$, which can be continued for all values of time $t \geqslant t_0$. Let us denote by $X[t_0, x_0]$ the set of solutions of (2.1) emerging from the initial point $\{t_0, x_0\}$ and defined on the segment $[t_0, \vartheta]$, i.e.

$$X[t_0, x_0] = \{x(t) \colon x(t) = x(t/t_0, x_0), \ t_0 \leqslant t \leqslant \vartheta\}$$

Here the solutions are by definition /1–3/ absolutely continuous functions of $x(t)$ for almost all t from the interval $[t_0, \vartheta]$, and satisfy the inclusion $x^{\cdot}(t) \in R(t, x(t))$.

The cross-section of the set $X[t_0, x_0]$ at $t = \vartheta$ will be called the domain of accessibility of system (2.1) from the state $x(t_0) = x_0$ to the instant of time $t = \vartheta$, and will be denoted by $G(\vartheta, t_0, x_0)$. Since condition 3) ensures that the set $X[t_0, x_0]$ is convex, it follows that when the conditions 1)–3) hold, the domain of accessibility $G(\vartheta, t_0, x_0)$ is a convex, closed and bounded set.

In order to give an analytic description of the domain of accessibility $G(\vartheta, t_0, x_0)$, we shall consider the following problem.

Problem 2.1. Let l be an arbitrary vector. We require to select, from the motions $x(t) \in X[t_0, x_0]$, a motion $x^{(0)}(t)$, such that

$$l' x^{(0)}(\vartheta) = \max_{x(t) \in X[t_0, x_0]} l' x(\vartheta) = \rho [l, \vartheta, t_0, x_0] \tag{2.4}$$

The motion $x^{(0)}(t) = x^{(0)}(t/t_0, x_0, l)$, which solves problem 2.1 and the control $w^{(0)}(t) = x^{\cdot(0)}(t)$, will be called optimal. The function $\rho [l, \vartheta, t_0, x_0]$ is a reference function of the domain of accessibility $G(\vartheta, t_0, x_0)$.

The sufficient conditions of optimality of the control $w^{(0)}(t)$ and the motion $x^{(0)}(t)$, expressed as a maximum principle, the following form:

Theorem 2.1. Let the conditions 1)–4) hold and let

$$\{x^\circ(t), \psi^\circ(t)\} = \{x^\circ (t/t_0, x_0, l), \psi^\circ (t/t_0, x_0, l)\}$$

be a solution of problem (2.3) for the initial position $\{t_0, x_0\}$, Then $x^\circ(t) = x^{(0)}(t)$, i.e. $x^\circ(t)$ is the optimal motion solving problem 2.1.

The validity of Theorem 2.1 follows from the general assertions given in the monograph

/5/ where a detailed bibliography also appears of works dealing with the derivation of the conditions of optimality for the systems described by differential equations.

We can write the sufficient conditions of optimality in a different form, using the standard arguments of the method of dynamic programming.

Theorem 2.2. Let the conditions 1)−2) hold. If a continuously differentiable function $\varkappa(t, x)$ can be indicated for Eq.(2.1) as well as the set $W^\circ(t, x) \subset R(t, x)$, semicontinuous from above on the inclusion when the position $\{t, x\}$ is varied, such that

a) whatever the value of the vector $w(t, x) \in R(t, x)$, the following inequality holds for all t and x

$$\frac{\partial \varkappa(t, x)}{\partial t} + w'(t, x) \frac{\partial \varkappa(t, x)}{\partial x} \leqslant 0$$

b) whatever the value of the vector $w^\circ(t, x) \in W^\circ(t, x)$, the following identity is satisfied for all t and x

$$\frac{\partial \varkappa(t, x)}{\partial t} + w^{\circ\prime}(t, x) \frac{\partial \varkappa(t, x)}{\partial x} \equiv 0$$

c) $\varkappa(\vartheta, x) = l'x$,

then any solution $x^\circ(t)$, $t_0 \leqslant t \leqslant \vartheta$, $x^\circ(t_0) = x_0$ of the equation in contingencies $x^\cdot \in W^\circ(t, x)$ will represent the optimal motion solving problem 2.1. The following relation also holds:

$$\varkappa(t_0, x_0) = \rho[l, \vartheta, t_0, x_0] = \max_{x(t) \in X[t_0, x_0]} l'x(\vartheta) \tag{2.5}$$

It follows, that in order to find a solution of problem 2.1 it is sufficient to find a continuously differentiable function $\varkappa(t, x)$, satisfying the equation

$$\frac{\partial \varkappa(t, x)}{\partial t} + \max_{w \in R(t, x)} w' \frac{\partial \varkappa(t, x)}{\partial x} = 0 \tag{2.6}$$

provided that

$$\varkappa(\vartheta, x) = l'x \tag{2.7}$$

To illustrate this, we shall consider a linear control system, i.e. we will assume that the set $R(t, x)$ is described by the equation

$$R(t, x) = \{w\colon w = A(t)x + B(t)u,\ u \in P(t)\}$$

where u is an r-dimensional vector taking values from the convex, closed and bounded set $P(t)$, $A(t)$, $B(t)$ are matrices of corresponding dimensions. The reference function of the domain of accessibility from any position $\{t, x\}$ has the form /1/

$$\rho[l, \vartheta, t, x] = l'Y[\vartheta, t]x + \int_t^\vartheta \max_{u \in P(\tau)} l'Y[\vartheta, \tau] B(\tau) u \, d\tau \tag{2.8}$$

where $Y[t, \tau]$ is the fundamental matrix of solutions of the linear homogeneous system $y^\cdot = A(t)y$. The function $\rho[l, \vartheta, t, x]$ (2.8) is continuously differentiable and is a solution of (2.6) with condition (2.7).

In the general case the sufficient conditions for the function $\varkappa(t, x)$, solving the Cauchy problem (2.6), (2.7) to exist, are given by the following theorem.

Theorem 2.3. If the conditions 1)−5) hold, then the reference function $\rho[l, \vartheta, t, x]$ of the domain of accessibility $G(\vartheta, t, x)$ of system (2.1) is continuously differentiable in t and x, and satisfies Eq.(2.6) with condition (2.7).

Proof. It is sufficient to show that the reference function $\rho[l, \vartheta, t, x]$ is continuously differentiable in x. Let us consider a convergent sequence of points $\{x_0^{(k)}, k = 1, 2, \ldots\} \to x_0$, and place in correspondence with it the sequence $\{x^\circ(t/t_0, x_0^{(k)}, l), k = 1, 2, \ldots\}$ of optimal motions and the sequence of functions $\{\psi^\circ(t/t_0, x_0^{(k)}, l), k = 1, 2, \ldots\}$, representing, at every $x_0^{(k)}$, a unique solution of problem (2.3). The sequences are uniformly bounded and continuous to the same power, and hence, by virtue of the uniqueness of the solution of (2.3), the following limit relations hold uniformly in $t \in [t_0, \vartheta]$:

$$\lim_{k\to\infty} x^\circ(t/t_0, x_0^{(k)}, l) = x^\circ(t/t_0, x_0, l) \tag{2.9}$$

$$\lim_{k\to\infty} \psi^\circ(t/t_0, x_0^{(k)}, l) = \psi^\circ(t/t_0, x_0, l)$$

Let us consider the solutions $x(t/t_0, x_0^{(k)} + y, l)$, $t_0 \leqslant t \leqslant \vartheta$ of the system of differential equations

$$x^\cdot = w^\circ(\psi^\circ(t/t_0, x_0^{(k)}, l), t, x), \quad k = 1, 2, \ldots \tag{2.10}$$

with initial conditions $x(t_0) = x_0^{(k)} + y$, $\|y\| \leqslant \alpha$, where α is a sufficiently small positive number. We know /6/ that for sufficiently small α a solution of system (2.10) exists, is unique and continuously differentiable with respect to the initial vector $y^{(k)} = x_0^{(k)} + y$, $k = 1, 2, \ldots$; $y^{(0)} = x_0 + y$. Using the theorems /6/ on the differentiability of solutions of (2.10) with respect to the initial data of Eq.(2.9) we find that the following limit relation holds for sufficiently small α uniformly in t and y:

$$\lim_{k\to\infty} \frac{\partial x\,(t/t_0,\, x_0^{(k)} + y,\, l)}{\partial y^{(k)}} = \frac{\partial x\,(t/t_0,\, x_0 + y,\, l)}{\partial y^{(0)}} \tag{2.11}$$

The method of choosing the function $w^\circ(\psi, t, x)$ implies that the following inclusion holds for any value of k:

$$x\,(t/t_0,\ x_0^{(k)} + y,\ l) \in X\,[t_0,\ x_0^{(k)} + y] \tag{2.12}$$

Let us introduce the functions

$$\varkappa^{(k)}\,(t_0,\ x_0^{(k)} + y) = l'x\,(\vartheta/t_0,\ x_0^{(k)} + y,\ l)$$

which are defined in some sufficiently small neighbourhood of the point $x_0^{(k)}$, $k = 1, 2, \ldots$ are differentiable in y for $\|y\| \leqslant \alpha$ and by virtue of their construction satisfy the relations

$$\lim_{y\to 0} \varkappa^{(k)}\,(t_0,\ x_0^{(k)} + y) = l'x^\circ\,(\vartheta/t_0,\ x_0^{(k)},\ l) = \varkappa\,(t_0,\ x_0^{(k)}) \tag{2.13}$$

Moreover, from inclusion (2.12) and the definition of the function $\varkappa\,(t_0, x_0)$ (2.5) it follows that the following inequality holds for $\|y\| \leqslant \alpha$:

$$\varkappa^{(k)}\,(t_0,\ x_0^{(k)} + y) \leqslant \varkappa\,(t_0,\ x_0^{(k)} + y) \tag{2.14}$$

Using the relations (2.11)–(2.14) and the arguments used in /7, p.1309/, we can show that the following relation holds:

$$\lim_{\Delta x_0\to 0} \frac{1}{\Delta x_0} \{\rho\,[l, \vartheta, t_0, x_0 + \Delta x_0] - \rho\,[l, \vartheta, t_0, x_0]\} = \left\{\frac{\partial\,[l'x\,(\vartheta/t_0,\, x_0 + y,\, l)]}{\partial y^{(0)}}\right\}_{y=0}$$

therefore the reference function $\rho\,[l, \vartheta, t, x]$ is continuously differentiable in x.

Using standard arguments of control theory we can confirm that the function $\varkappa\,(t, x) = \rho\,[l, \vartheta, t, x]$ is a solution of the Cauchy problem (2.6), (2.7).

3. Extremal strategies. Let the positions $y\,[t] = y$ and $z\,[t] = z$ be realized at some instant of time t. We shall assume that the conditions 1)–5) hold for (1.1) and denote by $\rho^{(1)}\,[l, \vartheta, t, y]$ and $\rho^{(2)}\,[l, \vartheta, t, z]$ the reference functions of the domains of accessibility $G^{(1)}(\vartheta, t, y)$ and $G^{(2)}(\vartheta, t, z)$ for the motions $y\,(\tau/t, y)$ and $z\,(\tau/t, z)$ (1.1), $t \leqslant \tau \leqslant \vartheta$ from the states $y\,(t) = y$ and $z\,(t) = z$ to the instant $\tau = \vartheta$.

We assume that the payoff function $\sigma\,(x) \geqslant 0$ of the game is convex and satisfies the global Cauchy-Lipschitz condition. Then the following relation holds /8/:

$$\sigma\,(x) = \max_{l\in L}\ \{l'x - \omega\,(l)\} \tag{3.1}$$

$$\omega\,(l) = \sup_{x\in R^n}\ \{l'x - \sigma\,(x)\},\quad L = \operatorname{dom}\,\omega\,(\cdot) = \{l \in R^n\colon\ \omega\,(l) < \infty\}$$

where $\omega\,(l)$ is a function, conjugate /5, 8/ to the convex function $\sigma\,(x)$.

Let us now introduce the maximin program quantity

$$\varepsilon^\circ\,(\vartheta,\ t,\ y,\ z) = \max_{z(\vartheta)\in G^{(2)}}\ \min_{y(\vartheta)\in G^{(1)}}\ \max_{l\in L}\ \{l'z\,(\vartheta) - l'y\,(\vartheta) - \omega\,(l)\}$$

According to the assertions proved in Sect.2 the domains of accessibility $G^{(1)}$ and $G^{(2)}$ are convex, closed and bounded, and the conjugate function $\omega\,(l)$ is convex. Therefore, using the general minimax theorem /9/ we can write

$$\varepsilon^\circ\,(\vartheta,\ t,\ y,\ z) = \max_{l\in L}\ \{\rho^{(2)}\,[l,\ \vartheta,\ t,\ z] - \rho^{(1)}\,[l,\ \vartheta,\ t,\ y] - \omega\,(l)\} \tag{3.2}$$

We will consider a regular case /1/ when the maximum on the right-hand side of (3.2) is attained, for all positions $\{t, y, z\}$, on the unique vector $l^\circ = l^\circ\,(\vartheta, t, y, z)$.

Definition 3.1. We shall call the strategies U_e and V_e the extremal strategies (ES) if they are determined, at every position $\{t, y, z\}$, by the sets $U_e^*\,(t, y, z)$ and $V_e^*\,(t, y, z)$, consisting of all vectors u_e and v_e which satisfy the conditions of maximum

$$\begin{gathered} u_e'\,\partial\rho^{(1)}\,[l^\circ, \vartheta, t, y]/\partial y = \max_{u\in P(t,\, y)}\ u'\,\partial\rho^{(1)}\,[l^\circ, \vartheta, t, y]/\partial y \\ v_e'\partial\rho^{(2)}\,[l^\circ, \vartheta, t, z]/\partial z = \max_{v\in Q(t,\, z)}\ v'\partial\rho^{(2)}\,[l^\circ, \vartheta, t, z]/\partial z \\ (l^\circ = l^\circ\,(\vartheta,\ t,\ y,\ z)) \end{gathered} \tag{3.3}$$

In regular case of ES U_e and V_e are admissible /1, 2/ and the following assertions hold.

Theorem 3.1. Let the game payoff $\sigma(x(\vartheta))$ (1.2) be a convex function satisfying the global Cauchy-Lipschitz condition, let conditions 1)−5) hold for Eqs.(1.1), and let it be the regular case. Then the ES U_e is the optimal strategy which solves problem 1.1. Moreover, we have

$$(\gamma[\vartheta]/t_0, y_0, z_0, U_e, v) \leqslant \varepsilon^\circ(\vartheta, t_0, y_0, z_0)$$

whatever the initial position $\{t_0, y_0, z_0\}$ and whatever the admissible realization $v[t]$ of the control v.

Theorem 3.2. Let the game payoff $\sigma(x, (\vartheta))$ (1.2) be a convex functions satisfying the global Cauchy-Lipschitz condition, let conditions 1)−5) hold for Eqs.(1.1), and let this be a regular case. Then the ES V_e is the optimal strategy which solves problem 1.2. Moreover,

$$(\gamma[\vartheta]/t_0, y_0, z_0, u, V_e) \geqslant \varepsilon^\circ(\vartheta, t_0, y_0, z_0)$$

whatever the initial position $\{t_0, y_0, z_0\}$ and whatever the admissible realization $u[t]$ of the control u.

To prove Theorems 3.1 and 3.2 consider the behaviour of the derivative $d\varepsilon^\circ[t]/dt$ of the absolutely continuous function $\varepsilon^\circ[t] = \varepsilon^\circ(\vartheta, t, y[t], z[t])$ along the motions $y[t]$ and $z[t]$ (1.1) generated by the strategies U_e, V and U, V_e.

We know that in the regular case the right-hand side of (3.2) is continuously differentiable in t, y and z. In computing the derivatives the dependence of the vector l° on the position $\{t, y, z\}$ is ignored; therefore the following relation holds:

$$d\varepsilon^\circ[t]/dt = \partial\rho^{(2)}[l^\circ, \vartheta, t, z]/\partial t + v'[t]\,\partial\rho^{(2)}[l^\circ, \vartheta, t, z]/\partial z - \partial\rho^{(1)}[l^\circ, \vartheta, t, y]/\partial t - u'[t]\,\partial\rho^{(1)}[l^\circ, \vartheta, t, y]/\partial y \tag{3.4}$$

$$(l^\circ = l^\circ(\vartheta, t, y, z),\ y = y[t],\ z = z[t])$$

From (3.4), Theorem 2.3 and the definition of ES it follows that $d\varepsilon^\circ[t]/dt \leqslant 0$ for almost all t, provided that the first player sticks to the ES and the second player uses any admissible strategy V. If on the other hand the second player sticks to the ES, V_e, and the first player uses any admissible strategy U, then $d\varepsilon^\circ[t]/dt \geqslant 0$ for all t, which proves Theorems 3.1 and 3.2.

Theorem 3.3. Let the game payoff $\sigma(x(\vartheta))$ (1.2) be a convex function satisfying the global Cauchy-Lipschitz condition, let the conditions 1)−5) hold for (1.1), and let this be a regular case. Then the ES U_e and V_e furnish a saddle point to the game-theoretic problem of aiming, whatever the initial position $\{t_0, y_0, z_0\}$.

4. Example. Let us consider the problem of the approach of quasilinear objects in the interval $[t_0, \vartheta]$, when the sets $P(t, y)$ and $Q(t, z)$ describing the domains of control of the players have the form

$$P(t, y) = \{u\colon \|u\| \leqslant \mu + \lambda\|y\|^2\},\quad Q(t, z) = \{v\colon \|v\| \leqslant \nu + \lambda\|z\|^2\} \tag{4.1}$$

where $\mu \geqslant \nu > 0, \lambda$ is a small parameter and the function $\sigma(x)$, determining the payoff of the game is given by the equation

$$\sigma(x) = \|z - y\| \tag{4.2}$$

and hence the quantity $\gamma[\vartheta]$ (1.2) determines the Euclidean distance between the objects at the given instant of time $t = \vartheta$.

Carrying out the necessary reduction we obtain

$$\rho^{(1)}[l, \vartheta, t, y, \lambda] = l'y + \mu\|l\|(\vartheta - t) + \lambda[\|l\|\|y\|^2(\vartheta - t) + \frac{1}{3}\mu^2\|l\|(\vartheta - t)^3 + \mu l'y(\vartheta - t)^2] + \ldots$$

$$\rho^{(2)}[l, \vartheta, t, z, \lambda] = \text{idem}\,\{\mu \to \nu,\ y \to z\}$$

In this case Eq.(3.2) has the form

$$\varepsilon^\circ(\vartheta, t, y, z, \lambda) = \max_{l\in L}\{\rho^{(2)}[l, \vartheta, t, z, \lambda] - \rho^{(1)}[l, \vartheta, t, y, \lambda]\},\quad L = \operatorname{dom}\omega(\cdot) = \{l\colon \|l\| \leqslant 1\} \tag{4.3}$$

When $\mu \geqslant \nu > 0$ the maximum on the right-hand side of (4.3) will be attained on the unique vector $l^\circ(\vartheta, t, y, z, \lambda)$ only in the region $\varepsilon^\circ > 0$. Therefore, following /1, 2/ we determine the ES U_e and V_e as follows: in the region $\varepsilon^\circ > 0, t < \vartheta$ the sets $U_e{}^*$ and $V_e{}^*$ are composed of all the vectors u_e and v_e which satisfy the condition of maximum (3.3), and in the region $\varepsilon^\circ = 0, t < \vartheta$ we will assume that $U_e{}^* = P$ and $V_e{}^* = Q$. Having carried out the necessary reduction, we find that the ES of the first and second player are given by the relations:

a) in the region $\varepsilon^\circ > 0, t < \vartheta$ the sets $U_e{}^*(t, y, z, \lambda)$ and $V_e{}^*(t, y, z, \lambda)$ consists of a single point

$$u_e[t] = (\mu + \lambda\|y\|^2)\frac{\partial\rho^{(1)}/\partial y}{\|\partial\rho^{(1)}/\partial y\|} = \mu l^\circ + \lambda\left[x\left(\frac{\|y\|^2}{\|x\|} - \mu\frac{x'y(\vartheta - t)}{\|x\|^3}\right) + 2\mu y(\vartheta - t)\right] + \ldots$$

$$v_e[t] = \text{idem}\{\mu \to \nu, y \to z, \rho^{(1)} \to \rho^{(2)}\}$$

b) if $\varepsilon^\circ = 0, t < \vartheta$, then

$$U_e^*(t, y, z, \lambda) = P(t, y), \quad V_e^*(t, y, z, \lambda) = Q(t, z)$$

The vector $l^\circ(\vartheta, t, y, z, \lambda)$, which furnishes a maximum to the right-hand side of Eq.(4.3), has the form

$$l^\circ = \frac{x}{\|x\|} + \lambda \frac{(\vartheta - t)^2}{\|x\|^3} [\|x\|^2 (\nu z - \mu y) + x(-\nu x'z + \mu x'y)] + \ldots$$

The ES U_e and V_e determined in this manner furnish the approach game (4.1), (4.2) with a saddle point, and the game payoff $\varepsilon^\circ(\vartheta, t, y, z, \lambda)$ is given, for any position $\{t, y, z\}$, by the equation

$$\varepsilon^\circ = \|x\| - (\mu - \nu)(\vartheta - t) + \lambda [(\|z\|^2 - \|y\|^2)(\vartheta - t) +$$
$$\|x\|^{-1} x'(\nu z - \mu y)(\vartheta - t)^2 + \frac{1}{3}(\nu^2 - \mu^2)(\vartheta - t)^3] + \ldots$$

REFERENCES

1. KRASOVSKII N.N., Game-Theoretic Problems of the Encounter of Motions. Moscow, Nauka, 1970.
2. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games. Moscow, Nauka, 1974.
3. FILIPPOV A.F., Differential equations with a discontinuous right-hand side. Mat. sb., 51, 1, 1960.
4. AL'BREKHT E.G., On the optimal control of motions of quasilinear systems. Differents. uravneniya. 5, 3, 1969.
5. PSHENICHNYI B.N., Convex Analysis and Extremal Problems. Moscow, Nauka, 1980.
6. STEPANOV V.V., Course of Differential Equations. Moscow, Fizmatgiz, 1959.
7. AL'BREKHT E.G., On the existence of an optimal Lyapunov function and a continuous optimal control for a problem on the analytic construction of regulators. Differents. uravneniya, 1, 10, 1965.
8. ROCKAFELLAR R.T., Convex Analysis. Princeton, N.Y. Princeton University Press, 1970.
9. KARLIN S., Mathematical Methods in the Theory of Games. Programming and Economics. London, Pergamon Press, 1959.

Translated by L.K.

PMM U.S.S.R., Vol.50, No.3, pp.260-265, 1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

THE APPLICATION OF MONOMIAL LIE GROUPS TO THE PROBLEM OF ASYMPTOTICALLY INTEGRATING EQUATIONS OF MECHANICS*

V.F. ZHURAVLEV

The basis of the algorithm of the asymptotic integration of equations of mechanics discussed below is the representation of the initial system as a monomial Lie group of transformations of the phase space into itself. Transformations of the system which reduce it to a simpler form are also sought in a class of systems possessing group properties. Matching the instrument of the analysis to the objective of the analysis enables us to limit the operations used in the algorithm to those from the corresponding operator algebra.

Hori's paper /1/, in which Lie series were used to construct an additional first integral in an autonomous Hamiltonian system, was followed by a number of papers which extended this approach to autonomous systems of general form (Hori, Kemel et al, a review of whose results can be found in /2, 3/). Note that all these papers are essentially only different forms of deriving Hausdorff's formula, which is well-known from the theory of Lie groups, complicated somewhat by the concept of parameter identification and order separation. Now results can only be obtained by refusing to consider systems of general form and by proceeding to

*Prikl.Matem.Mekhan., 50, 3, 346-352, 1986